Advances in Computational Limit State Analysis and Design

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ABSTRACT

While simple closed form limit analysis solutions continue to be widely used in geotechnical engineering practice, many practitioners are unaware that recently developed computational limit analysis methods now allow direct analysis of the collapse (or 'ultimate limit') state for problems of arbitrary geometry, obviating the need to resort to considerably more complex alternative techniques, such as non-linear finite elements.

This paper provides an overview of the state-of-the-art in the field of computational limit analysis, focusing in particular on the groundbreaking 'Discontinuity Layout Optimization' (DLO) numerical procedure. In this paper the DLO procedure is applied to a wide range of geotechnical problems, including those involving soil reinforcement and seismic loading, and it is shown that this technique may easily be integrated with LFRD limit state design methods.

INTRODUCTION

For several decades there has been a gap in the range of analysis tools available to practitioners, who have had to rely on simple hand analysis methods, perhaps automated and embedded in a spreadsheet or simple software application, or on complex methods such as Non-Linear Finite Element Analysis (NLFEA). However, computational limit analysis methods offer the possibility of bridging this gap, allowing the engineer to directly and straightforwardly determine the collapse (or ultimate limit) state for any geometry of problem, without the need to resort to NLFEA and comparable methods.

At the same time there is also a general shift towards the adoption of Limit State Design (LSD) principles, which explicitly require determination of the ultimate limit state (ULS). This is exemplified by the fact that during 2010 LSD is due to be implemented across Europe via adoption of the 'Eurocode 7' code of practice (BSI, 2004).

This paper describes the latest developments in the application of computational limit analysis and design techniques in geotechnical engineering practice. Currently available computational limit analysis methods are discussed, including the powerful new numerical analysis procedure, Discontinuity Layout Optimization (DLO) which was recently co-developed by the authors, and is now incorporated in the LimitState:GEO software application which is available for use by practitioners and academics (LimitState, 2009).

The broad range of applicability of computational limit analysis tools will be illustrated through practical examples, including those involving bearing capacity, slope stability, retaining structures, reinforced soils and also seismic loading.

The general application of computational limit analysis in design requires that it should be readily usable in conjunction with methods such as Load and Resistance Factor Design (LRFD). While at first sight a challenging task, this paper outlines how LRFD can be used in conjunction with computational limit analysis methods.

COMPUTATIONAL LIMIT ANALYSIS AND DESIGN METHODS

Limit analysis vs limit equilibrium methods

Limit analysis is a direct method of analysis used to find the maximum load sustainable by a body or structure, which is assumed: (i) not to deform prior to collapse; (ii) to deform at constant load subsequently (i.e. a rigid-plastic idealization). Formal plasticity theorems state that a computed collapse load will be:

1. A 'lower bound' if equilibrium and yield conditions are satisfied everywhere;
2. An 'upper bound' if flow rule and compatibility conditions are satisfied.

Thus a lower bound requires determination of a stress field satisfying 1. everywhere in the domain, and an upper bound requires determination of a collapse mechanism satisfying 2. The maximum lower bound load (or 'load factor') will be equal to the minimum upper bound load, with both giving the exact plastic collapse load (or 'limit' load). One well known limit analysis method is the Coulomb wedge method for retaining walls, which provides upper-bound solutions.

Like limit analysis methods, *limit equilibrium* methods are also direct methods of analysis. However, the formal theorems of plasticity are not strictly enforced (e.g. the postulated mechanism need not be kinematically admissible), which means that limit equilibrium methods can generally provide only approximate estimates of collapse load, with no indication as to whether this is higher or lower than the exact plastic collapse load. Though often considered to be a lower-bound limit analysis method, in a Rankine retaining wall analysis, equilibrium and yield conditions are not necessarily satisfied everywhere in the domain; thus a Rankine analysis should strictly speaking be considered as a limit equilibrium analysis method.

A distinct advantage of both limit analysis and limit equilibrium analysis methods is that they typically only require material strength (e.g. undrained shear strength, c_u; drained angle of shearing resistance, ϕ') and unit weight (γ) input parameters.

Automated hand calculation methods

A simple retaining wall analysis can for example be carried out using the Coulomb wedge method, using one or more wedges, as depicted in Figure 1. Such an analysis is relatively easy to carry out by hand (or using a spreadsheet), but only when simple geometries are involved. Usually numerical or analytical optimization techniques are used to determine the wedge angles corresponding to the critical collapse mechanism. The problem typically becomes quite challenging when more than one or two wedges are involved.

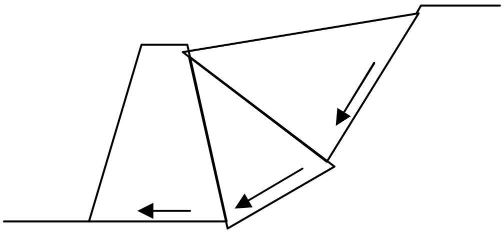


Figure 1. Two wedge Coulomb analysis

More sophisticated slope stability software can carry out 'slip circle' (limit analysis or limit equilibrium) or 'method of slices' (limit equilibrium) analyses. However since only a restricted range of collapse mechanisms are considered, the software cannot be considered to be generally applicable.

Method of characteristics

This *method of characteristics* solution procedure is presented in detail in the classic 'Statics of Granular Media' text by Sokolovskii (1965). The method assumes that all or part of the soil domain in question is in state of yield and solves the corresponding partial differential equations for the soil stresses that result from this assumption (using the well known 'method of characteristics' mathematical procedure). The characteristic lines that result represent surfaces on which the soil yields. A typical solution for a bearing capacity problem is shown in the grey shaded area in Figure 2. On its own this provides a limit equilibrium solution of indeterminate status. However if an equilibrium stress field that does not violate yield can be extended over the whole problem domain, a true lower bound solution is found, as is shown for example in the unshaded area in Figure 2 (Smith, 2005).

While a very powerful tool for specific problems, it has not yet been found possible to generalize it or to straightforwardly handle problems with mixed areas of yielding and non yielding soil. A good example of a numerical implementation of the method for the solution of bearing capacity problems can be found in Martin (2004). The ABC software developed for this purpose is able to solve both plane strain and axisymmetic problems.

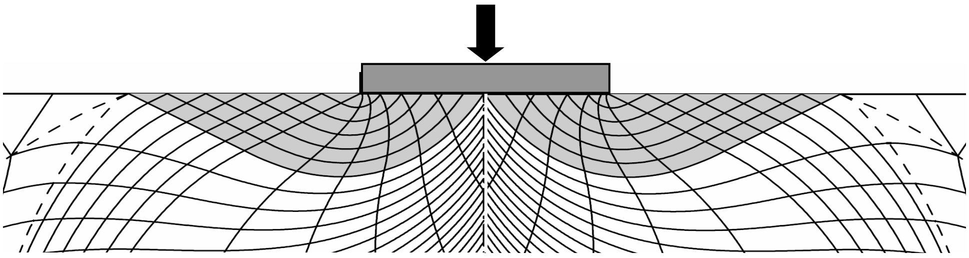


Figure 2. Method of characteristics solution for a smooth footing on a frictional soil (the 'characteristic' lines represent surfaces on which yield occurs)

Finite element limit analysis

When used to solve conventional linear-elastic problems, finite element analysis involves the solution of a set of linear equations. If material non-linearity is introduced a sequence of such equations can be set up and solved, as part of an iterative solution procedure. However, finite elements can also be used in conjunction with a rigid-plastic constitutive model, leading to a potentially much simpler optimization problem which can be solved directly, without the need to iterate. Potential benefits of such a 'finite element limit analysis' approach include the need for far fewer input parameters and obviation of the need to consider the initial stress state; these benefits have long been recognized by researchers, with notable contributions relevant to the field of geomechanics including those of Lysmer (1970), Sloan (1988) and Makrodimopoulos & Martin (2006). In the literature a wide range of applications have been considered, e.g. involving: 2D and 3D bearing capacity, slope stability (incl. rock slopes and seismic loads), retaining wall stability, anchor uplift, etc. Advantages of the method include:

- General applicability and potential to obtain high accuracy.
- Ability to provide upper and/or lower bound solutions.
- 3D problems can be solved (though are challenging at present).

Conversely, disadvantages of the method include:

- The frequent need for a specially tailored mesh (or complex adaptive mesh refinement procedure), as singularities in the displacement and/or stress fields can be challenging.
- Graphical output is not always easy to interpret.
- No readily usable tool available to practitioners has to date been developed (i.e. currently only specialist/research codes are available).

Commentary

The failure of finite element limit analysis to find use in industry at first sight appears both surprising and regrettable. However it is evident that the method does share a number of the problems encountered when using conventional NLFEA (e.g. mesh sensitivity near singularities, difficulty validating results etc), and also lacks the inherent simplicity of some of the traditional hand-based limit analysis methods (e.g. the traditional 'upper bound' hand analysis method). With this firmly in mind, the authors have recently developed an alternative numerical procedure, Discontinuity Layout Optimization (DLO), which will now be described.

DISCONTINUITY LAYOUT OPTIMIZATION (DLO)

DLO is a recently developed computational limit analysis procedure which involves use of optimization techniques to identify the critical layout of discontinuities (slip-lines) in a failing soil mass (Smith & Gilbert, 2007). In essence the method is able to determine the critical slip-line collapse mechanism for any problem geometry. Figure 4 indicates how a problem is set up and solved using the procedure.

As implied on the figure, the problem is couched in terms of potential discontinuities which inter-link nodes used to discretize the body of soil being analysed. In the general case each node is connected to every other node by a potential discontinuity. In the kinematic formulation it is only necessary to enforce compatibility at nodes to ensure a compatible mechanism. (When discontinuities crossover one another, compatibility at 'crossover points' can be shown to be enforced implicitly.) The objective of the optimization process is to find the minimum upper-bound solution. In effect the optimization process selects the subset of discontinuities required to form the critical collapse mechanism from the starting set of all potential discontinuities. Increasing the number of nodes increases the accuracy of the solution obtained.

The DLO procedure is now incorporated in the LimitState:GEO software application which is available for use by practitioners and academics (LimitState, 2009).

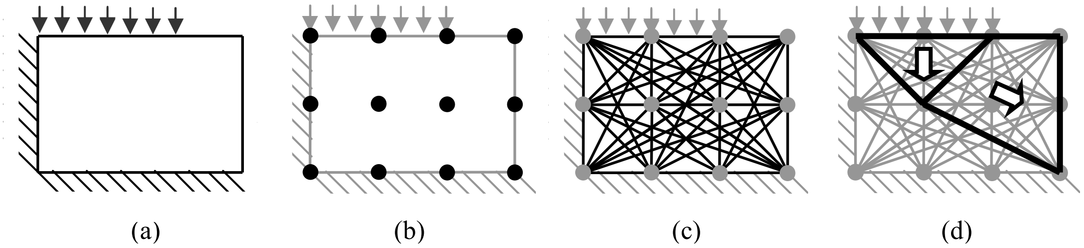

(a) (b) (c) (d)

Figure 4. Stages in DLO procedure: (a) starting problem (surcharge close to vertical cut); (b) discretization of soil using nodes; (c) interconnection of nodes with potential discontinuities; (d) identification of critical subset of potential discontinuities using optimization (giving the critical failure mechanism)

Advantages of the DLO procedure include:

- Generally applicable and potential to obtain high accuracy (linear optimization process guarantees identification of a global optimum).
- Handles singularities inherently.
- No advance knowledge of the solution is required.
- An easy-to-use software tool is now available for use by practitioners.
- The output is easy to interpret.

Conversely, limitations of the DLO procedure include:

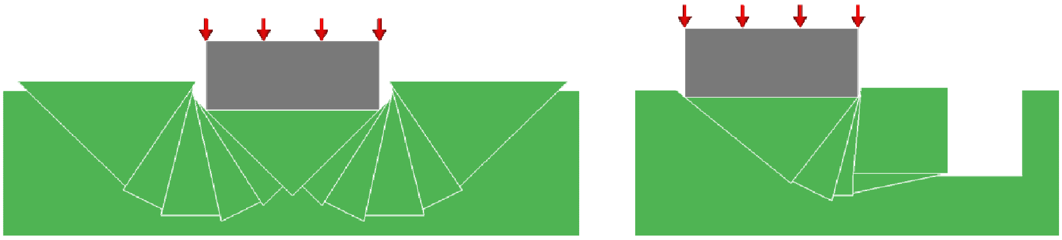
- Upper bound only at present (though the high accuracy solutions obtained will often be close to 'true' solutions).
- 2D plane strain only at present.

COMPUTATIONAL LIMIT ANALYSIS EXAMPLES

The range of applicability of the DLO procedure described previously will now be illustrated through application to a range of example problems.

Simple foundation footing problems

Figure 5 shows sample DLO output for two simple foundation footing problems. In both cases a target of 500 nodes was specified and the results were obtained in a matter of seconds on a modern desktop PC.



(a) (b)

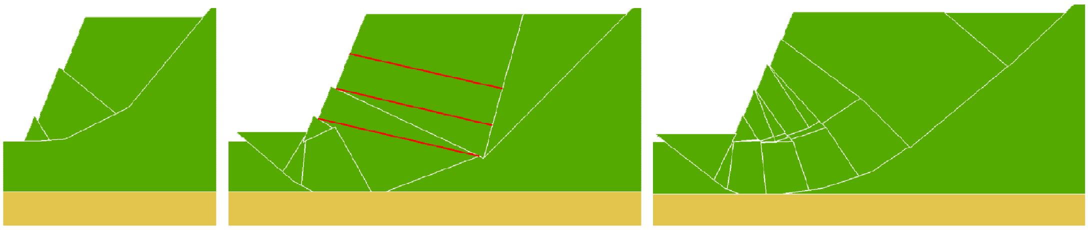
Figure 5. Foundation failure mechanisms: (a) 1m wide footing on weightless undrained clay of unit cohesive strength (Prandtl problem), solution = 5.19; (b) as (a) but with 0.5x0.5m trench 0.5m from edge of footing, solution = 3.70

The solution for the problem shown in Figure 5(a) of 5.19 is within 1% of the known analytical solution for this problem (2+π). It is evident that the maximum bearing pressure reduces significantly when a trench is excavated close to the footing (solution falls from 5.19 to 3.70). Although the new critical failure mechanism is relatively simple, it would be time-consuming for most engineers to identify by hand, and can in contrast be found very rapidly using the DLO software.

DLO results for footing problems involving frictional soil are provided in Smith & Gilbert (2007).

Simple slope stability problems

Figure 6 illustrates how the DLO procedure can be used to investigate the influence of features such as soil nails, Figure 6(b), and horizontal seismic loading, Figure 6(c), on the extent of the failure mechanism in a slope stability problem.



(a) (b) (c)

Figure 6. Slope problems: (a) undrained clay slope; (b) as (a) with soil nails; (c) as (a) but subject to seismic loading (using pseudo-static modelling approach)

Construction plant at crest of a layered slope

Civil engineering contractors are routinely required to identify safe routes for construction plant. Whereas this has typically involved application of experience, rules of thumb and/or very simple hand type calculations, the availability of modern computational limit analysis techniques mean that it is now possible to set up and solve an appropriate model in a matter of minutes. Figure 7 shows sample output (for the case of an excavator with outriggers on a layered undrained clay soil).

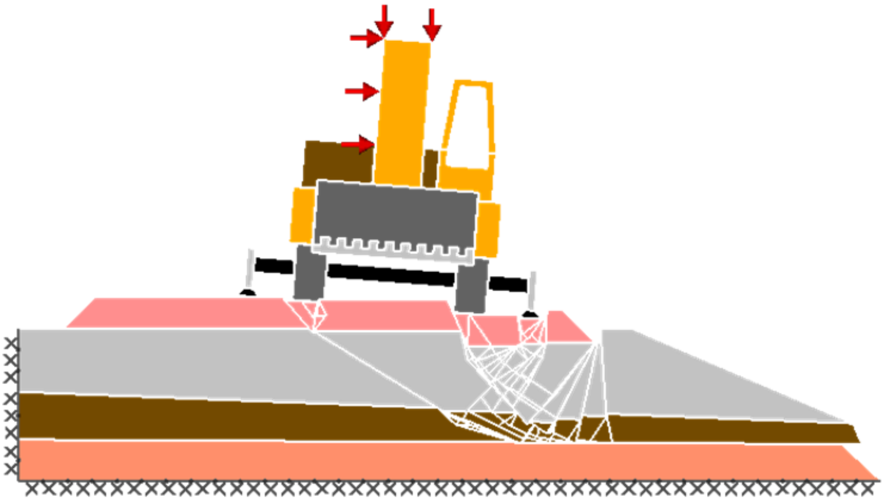


Figure 7. Construction plant at the crest of a layered slope leading to a rotational failure (courtesy of LimitState Ltd)

USE OF COMPUTATIONAL LIMIT ANALYSIS WITH LRFD

In Load and Resistance Factor Design (LRFD), it is necessary to apply factors to applied loads and to soil resistance. If an attempt is made to directly apply these factors as part of a computational limit analysis procedure, then it is typically found that internal equilibrium is violated and that the process will fail.

For simple problems (e.g. involving bearing capacity of a surface footing) this problem does not arise since the factors are effectively applied at an interface between an externally applied load and the soil/structure domain. Therefore lack of internal equilibrium is not encountered. However, when applied for example to retaining wall problems, where the factors may be applied between the soil mass and the wall, the direct approach does not work and it is necessary to adopt an alternative procedure.

It is first instructive to consider how a conventional geotechnical analysis is undertaken: In general a given design will be inherently stable and purposely far from its ultimate limit state. In order to undertake a ULS analysis it is necessary to drive the system to collapse by some means. More often that not in a conventional analysis this is undertaken implicitly. However, when using a general numerical analysis tool this must be done explicitly.

In a conventional LRFD approach it is implicitly assumed that the soil is in a state of collapse around the foundation/structure. For example, in the ULS design of the gravity wall depicted in Figure 8 against sliding or overturning, active and passive Rankine pressure distributions are typically assumed to act on each side of the wall, without any rigorous theoretical justification.

The LRFD design check requires that:

$$LF \times A \ < \ RF_S \times S + RF_P \times P \qquad (1)$$

Where LF is the load factor and RF_S and RF_P are the resistance factors for sliding and passive pressure respectively. This is in contrast to considering the FOS as the ratio $R = A \,/\, (P + S)$ of resisting forces to disturbing forces.

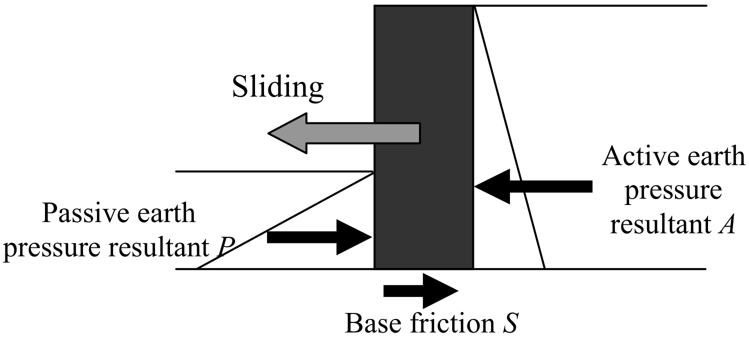


Figure 8. Conventional assumption made in the analysis of a gravity retaining wall against sliding.

The challenge when using this approach in conjunction with a computational limit analysis procedure arises because the above system is not in equilibrium if $R \neq 1$. For example if $R > 1$ then the passive earth pressure and base friction significantly exceed the active earth pressure. In reality equilibrium occurs because the stress distributions typically do not reach their failure conditions; however in a ULS analysis this is required.

The solution to this challenge is to preserve the equilibrium in the ULS numerical analysis by applying a 'hypothetical' external force H in the direction of assumed failure, and to then increase this force until failure occurs. This is a straightforward calculation for a computational limit analysis method, and ensures equilibrium is preserved at all times.

Once failure has been achieved, appropriate 'passive' and 'active' zones are generated on either side of the wall, as assumed in conventional analysis. It is then straightforward to determine the active force (A) and passive and base resistance (P and S) predicted by the analysis, and to incorporate these into the LRFD equation (1). H itself is ignored.

The disadvantage of the LRFD approach is that the mode of failure must be pre-determined in this method and the LRFD equation can only be applied in the sense of the imposed failure mode. i.e. it would not be permitted to use the values computed to make an assessment against overturning; instead a hypothetical moment M would have to be applied to the wall until failure occurred with the LRFD equation formulated in terms of moments. However the soil resistance would be correctly modelled for the specified failure mode. Note that the assumption of a Rankine distribution of soil pressures is in this case not necessarily correct.

An example of an LRFD check against a combined bearing/sliding failure of a wall is given in Figure 5. Here the externally applied force (H) acts at an angle of 45 degrees, as indicated. Components of the self weight (W) and the resultant loads

(*A*) and resistances (*P* and *S*) arising from the forced failure mechanism in this direction are given in Table 1. Fully frictional soil/wall interfaces are assumed. The solution shown was identified using the LimitState:GEO software (LimitState, 2009) and demonstrates how the loads and resistances on the wall can be determined by the application of the hypothetical external force on the wall in the direction of failure that is to be checked.

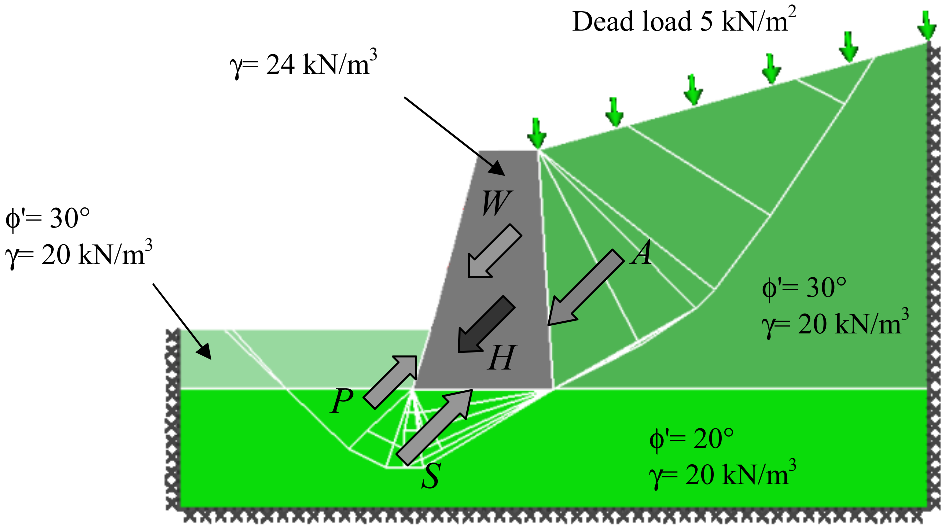


Figure 5. Design check against combined sliding and bearing capacity failure of a 4m high gravity retaining wall using LimitState:GEO (translational mode)

Table 1 indicates that when considering the factored loads and resistances in the direction of externally disturbing force, this wall is unsafe for the specific failure mode. One challenge for the LRFD method is identification of suitable factors for combined and other non-standard failure modes.

Table 1. Components of self weight (W) and resultant loads (*A*) and resistances (*P* and *S*) arising from the forced failure mechanism shown in Figure 5.

		Value	**Factor†**	**Product**
Loads (kN/m)	*W*	112.7	1.35	152.1
	A	75.8	1.35	102.3
	sum	188.5		254.4
Resistances (kN/m)	*P*	34.8	0.7	24.4
	S	203	0.7	142.1
	sum	237.8		166.5
Outcome		$L < R$ Safe		$L \times LF > R \times RF$ Unsafe

† factors used are illustrative and are not taken from a specific code of practice.

The LRFD method is similar to Design Approach 2 of Eurocode 7 (BSI, 2004). The alternative Eurocode Design Approach 1 avoids this problem by factoring the soil strength responsible for providing the resistance, rather than the resistance itself (i.e. the soil strength is factored at source). This provides computational limit analysis methods, and other general numerical analysis methods, with the freedom to determine the actual critical collapse mode without the user having to consider each possible mode in turn - with the risk that the critical mode might be missed.

CONCLUSIONS

1. Computational limit analysis techniques can bridge the gap between existing hand type calculation methods, which tend to have very narrow applicability, and significantly more complex tools such as non-linear finite element analysis (NLFEA).
2. The recently developed Discontinuity Layout Optimization (DLO) procedure for the first time allows direct automatic identification of critical collapse mechanisms in geotechnical problems of arbitrary geometry. An easy to use DLO-based analysis software tool is now available for use by practitioners.
3. A methodology that allows the use of LRFD in conjunction with computational limit analysis methods has been outlined.

ACKNOWLEDGEMENTS

The development of the DLO procedure has been supported by EPSRC (grant references GR/S53329/01 & GR/S53336/01). All DLO analyses described herein were undertaken using LimitState:GEO, see: http://www.limitstate.com/geo

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